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Shear layer instability of an inviscid compressible fluid. Part 2

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The linear stability of a shear layer of an inviscid compressible fluid is considered. It is shown that there is instability of two-dimensional disturbances at all values of the Mach number, contrary to previous results for a vortex sheet. The difference arises from the discovery of a second unstable mode. This mode is supersonic, decays weakly with distance from the shear layer, and is not governed by the principle of exchange of stabilities. Detailed numerical and asymptotic results are given for the hyperbolic-tangent shear layer.

1. Introduction

In 1970 Blumen found an analytical neutral curve and computed some stability characteristics for a smoothly varying shear layer of an inviscid perfect gas at uniform temperature. Only two-dimensional subsonic disturbances were considered and instability was shown to exist for $0 \leqslant M < 1$, where the Mach number M is based upon half the velocity difference across the shear layer. This is consistent with a classical argument attributed to Ackeret (by Liepmann & Puckett 1947, pp. 240–241), according to which the shear layer should be unstable only when $M < 1$. However, Drazin & Howard (1966, pp. 48–49, 57–58) had pointed out an apparent contradiction between this heuristic physical argument of Ackeret and some calculations of Landau (1944) and Hatanaka (1947), who found the criterion for instability of a vortex sheet to two-dimensional waves to be $M < 2^{\frac{1}{2}}$.

In this paper we shall resolve these apparent contradictions, showing that there is a second mode of instability of a smoothly varying shear layer of compressible fluid, that the curve of marginal stability is not as Blumen (1970) drew it, that the heuristic physical argument of Ackeret leads to a false conclusion, and that, indeed, the shear layer is unstable to two-dimensional waves at each value of the Mach number. The argument of Ackeret may be wrong because it applies the theory of steady flow to unsteady disturbances. The stability characteristics of a smoothly varying shear layer will be found to be much more intricate than was previously envisaged. The occurrence of the second mode is associated with a breakdown in the validity of the principle of exchange of stabilities.

This classical problem is of wider significance than its aeronautical origin. It has received attention recently because of the relevance of the stability of compressible flow to the generation of aerodynamic noise (see, for example, Howe 1970). Our results suggest that workers on aerodynamic noise should treat with caution models with vortex sheets, because the properties of long waves on smoothly varying shear layers differ significantly from those of waves on a discontinuous vortex sheet.

The mathematical intricacies of our solution are of more than technical significance also because similar intricacies occur in many stability problems with rotation or stratification rather than compressibility. Dickinson & Clare (1973) studied a problem of barotropic stability of incompressible fluid, for which they discovered a second mode of instability and an abrupt change in the direction of the curve of marginal stability, thereby revising the previous picture of the marginal curve. Huppert (1973) pointed out similar abrupt changes in marginal curves for flows of stratified incompressible fluid, and showed that the established formula for perturbation of neutral curves was sometimes invalid. Indeed, the apparent contradiction between the stability characteristics of a smoothly varying shear layer and a vortex sheet seem to recur for a stratified incompressible fluid (Drazin 1958; Drazin & Howard 1966, p. 46). Also Blumen (1975) has considered the stability to long waves of a non-planar shear layer in a vertically stratified incompressible fluid, for which we now suspect difficulties similar to those resolved in this paper. We believe that the ideas which we apply below to the stability of a shear layer in compressible fluid may, with suitable modifications, be applied to these other stability problems.

In this paper we shall treat only two-dimensional disturbances. Now Squire's transformation reduces the effective Mach number of a three-dimensional wave disturbance by a factor of the cosine of the angle the wave makes with the plane of the basic flow in giving the equivalent two-dimensional wave (Dunn & Lin 1952; Fejer & Miles 1963). Thus, in particular, a wave nearly perpendicular to the plane of compressible flow is equivalent to a two-dimensional incompressible wave, because the effective basic velocity is small. It follows that if a given flow is unstable to two-dimensional waves at zero Mach number, it is unstable to three-dimensional waves at each value of the Mach number. However, the criterion for stability of two-dimensional compressible waves determines, by Squire's transformation, the *direction* of unstable three-dimensional waves, an important matter to those who live or work near an airport.

In §2 we shall state the stability problem already posed. In §3 we shall summarize our numerical results, giving pictures of the marginal curve of stability. Many of these results are substantiated in §4 by asymptotic formulae, which give perturbations of the neutral curve and the long-wave approximation. These numerical and analytical results are related and discussed in the final section.

2. The eigenvalue problem

First we shall state the problem attacked by Blumen (1970). We consider the stability of the basic unbounded plane parallel flow with velocity

$$\mathbf{u} = U(y)\,\mathbf{i} \quad \text{for} \quad -\infty < y < \infty \tag{1}$$

of an inviscid perfect gas at uniform temperature. Then it follows (cf. Blumen 1970) that the stability of normal modes is governed by the equation

$$\{p'/M^2(U-c)^2\}' + \alpha^2\{1 - 1/M^2(U-c)^2\}p = 0 \tag{2}$$

and the boundary conditions

$$p \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{3}$$

Here we use dimensionless variables throughout, so that we may identify the Mach number M as the reciprocal of the speed of sound. The disturbance is supposed to be adiabatic, and its pressure to have the form $p(y)\exp\{i\alpha(x-ct)\}$ in terms of the positive wavenumber α and the complex velocity $c = c_r + ic_i$. Primes denote differentiations with respect to y. (The boundary conditions (3) should be replaced by radiation conditions for neutrally stable modes ($c_i = 0$), but this will not be necessary here because we shall treat only unstable ($c_i > 0$) and marginally stable (as $c_i \downarrow 0$) modes.)

Blumen (1970) considered the shear layer with

$$U = \tanh y \tag{4}$$

and discovered the neutral eigensolution

$$c = 0, \quad \alpha^2 + M^2 = 1, \quad p = (\operatorname{sech} y)^{\alpha^2}. \tag{5}$$

He concluded that this was a marginal eigensolution, with instability if and only if $\alpha^2 + M^2 < 1$. However, we shall show that there is instability just outside part of this circle in the α, M plane.

For the vortex sheet with

$$U = \pm 1 \quad \text{for } y \gtrless 0 \tag{6}$$

and uniform basic temperature, Landau (1944) and Hatanaka (1947) found an eigensolution which gives

$$\left.\begin{aligned} c^2 &= 1 + M^{-2} - M^{-2}(1+4M^2)^{\frac{1}{2}} \equiv c_v^2, \quad \text{say}, \\ p &= (c \mp 1)^2 \exp[\mp \alpha\{1 - M^2(c\mp 1)^2\}^{\frac{1}{2}} y] \quad \text{for} \quad y \gtrless 0. \end{aligned}\right\} \tag{7}$$

There is also a degenerate solution with

$$c = 0, \quad M^2 \geqslant 1, \quad p = \exp\{-i\alpha(M^2-1)^{\frac{1}{2}} y\}. \tag{8}$$

Note that solution (7) gives the criterion $M^2 > 2$ for stability of two-dimensional waves. Also, when $M^2 = 1$ the long-wave limit of Blumen's solution (5) as $\alpha \to 0$ is not solution (7), but (8).

c_i	M	c_r	M
0·00	1·4142	0·00	1·4142
0·01	1·4140	0·05	1·4195
0·03	1·4123	0·10	1·4356
0·05	1·4089	0·15	1·4630
0·07	1·4039	0·20	1·5023
0·09	1·3971	0·25	1·5549
0·10	1·3932	0·30	1·6225
0·20	1·3324	0·35	1·7075
0·30	1·2377	0·40	1·8133
0·40	1·1174	0·45	1·9461
0·50	0·9798	0·50	2·1082
0·60	0·8319	0·60	2·5769
0·70	0·6778	0·70	3·3848
0·80	0·5174	0·80	5·0308
0·90	0·3406	0·90	9·9861
1·00	0·0000	1·00	∞

TABLE 1. Unstable eigenvalues c_i and stable eigenvalues c_r for the vortex sheet given by (7)

3. Numerical results

If the variable $q = q_r + iq_i = p'/p$ and the transformation $z = \tanh y$ are introduced, (2) reduces to the form $q' = f(z, q)$ in $-1 \leqslant z \leqslant 1$. The functional form of $f(z, q)$, together with the appropriate boundary conditions, has been presented by Blumen (1970). As before, the shooting method was applied to solve this two-point boundary-value problem for the complex wave speed $c(\alpha, M)$. A step size $\Delta z = 0{\cdot}01$ was used to provide greater accuracy in the delineation of c than was previously obtained. Moreover, a complex quadratic interpolation scheme speeded up the rate of convergence to the correct eigenvalues.

Figures 1–4 display the distribution of the eigenvalues c in the α, M plane. Figures 5(*a*) and (*b*) are enlargements of parts of the α, M plane that enhance detail not clearly depicted in the previous diagrams. Note that the isolines of lower values of c_i in figures 2 and 5(*b*) cross isolines of higher values of c_i near $M = 2^{\frac{1}{2}}$ ($\alpha \doteqdot 0$) before each isoline intersects the M axis. The values of c_i along the M axis are then in agreement with the vortex-sheet solution given by (7) and displayed in table 1

Blumen assumed that $\alpha^2 + M^2 = 1$ was a neutral-stability curve in $0 \leqslant \alpha \leqslant 1$. Consequently, only the stationary unstable eigenvalues up to about $M = 0{\cdot}9$ were computed, under the assumption that $c_i \to 0$ as $M^2 \to 1 - \alpha^2$. As noted in the introduction, the present extension was undertaken in order to resolve the discrepancy between these previous computations and the known stability characteristics of the vortex sheet. The present computations, in conjunction with the asymptotic formulae presented in §4, have resolved the apparent discrepancy noted above. Moreover, two travelling unstable modes not delineated by the vortex-sheet solution were also discovered. The unstable eigenvalues associated with these latter solutions are shown in figures 3 and 4. These diagrams were

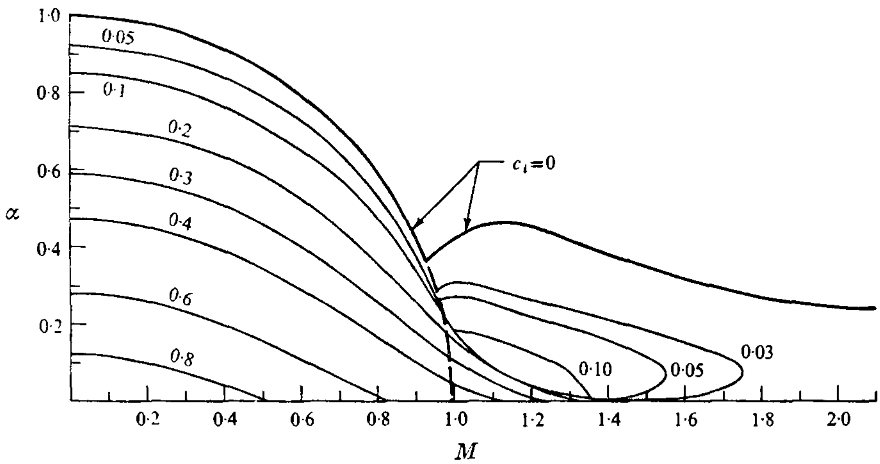

FIGURE 1. The unstable region in the α, M plane for the hyperbolic-tangent velocity profile. The thick solid lines, labelled $c_i = 0$, are curves of marginal stability. A stationary stable mode ($c_i = 0$) also exists along the thick dashed line that extends into the unstable region. The thin solid lines depict isolines of the imaginary part c_i of the complex phase speed.

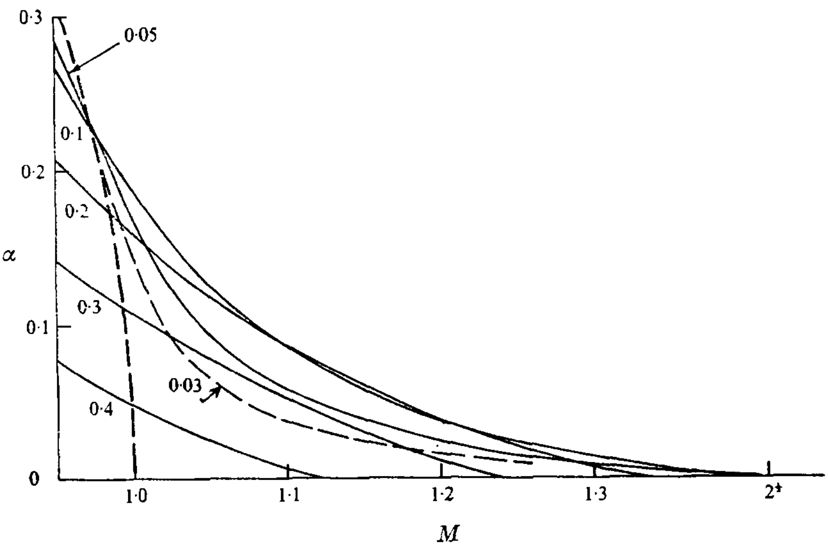

FIGURE 2. Imaginary phase speeds c_i in the unstable region of the α, M plane where c_i is double valued.

constructed for $c_r > 0$ but there is a corresponding mode, not shown, with $c_r < 0$. The stability boundary emanating from the curve $\alpha^2 + M^2 = 1$ seems to extend to $M = \infty$. Also, this stability boundary is not an exact representation. It was constructed from computations for $c_i \approx 5 \times 10^{-3}$, essentially the lowest value that could be determined by the present numerical technique.

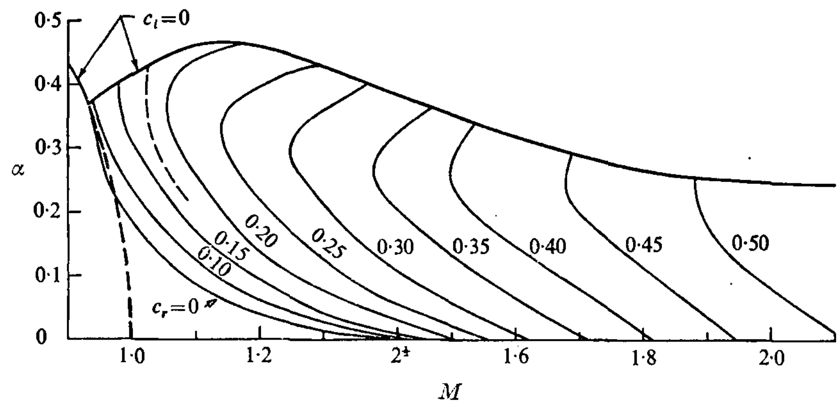

FIGURE 3. Real phase speeds c_r in the unstable region of the α, M plane.

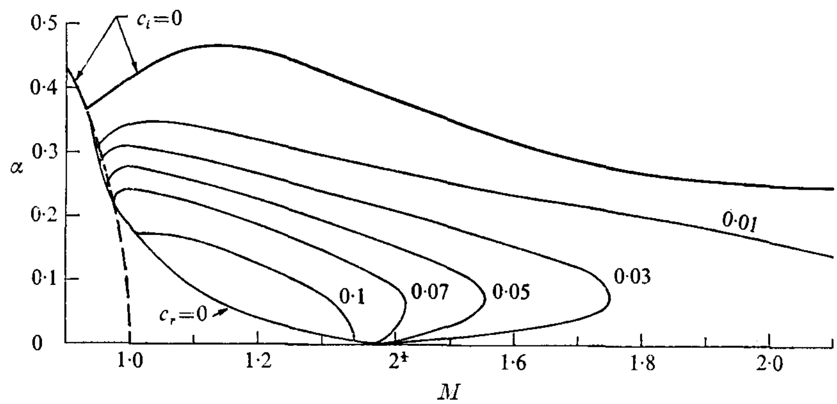

FIGURE 4. Imaginary phase speeds c_i in the unstable region of the α, M plane where $c_r \neq 0$.

Eigenfunction and Reynolds-stress computations in the region $\alpha^2+M^2 < 1$ have been presented by Blumen (1970). The eigenfunctions associated with both stationary modes ($c_r = 0$) in the region $\alpha^2+M^2 > 1$ are very similar in form to those interior to the quarter-circle and are not given here. The eigenfunctions associated with the travelling modes do not exhibit the symmetry properties associated with the stationary modes. The change in form that occurs is clearly exhibited by the initial Reynolds-stress distribution appearing in figure 6. The Reynolds stress averaged over one wavelength is denoted by $\tau = -\overline{\mathrm{Re}\,u\,\mathrm{Re}\,v}$, where expressions for the (x, y) velocity components (u, v) have been presented by Blumen (1970, equations 40 and 41). The computations of τ along $M = 1{\cdot}14$ show that τ is a symmetric function of y when $c_r = 0$ ($\alpha = 0{\cdot}06$, $0{\cdot}07$). In the region where travelling modes exist ($\alpha \gtrsim 0{\cdot}072$) the distribution of τ becomes increasingly asymmetric as the neutral-stability boundary is approached, tending towards a discontinuous change in τ near the origin.

It is known (Lin 1953) for a general velocity profile that, in the limit as $c_i \to 0$, τ is piecewise constant between critical layers where $U(y) = c$. For our monotone

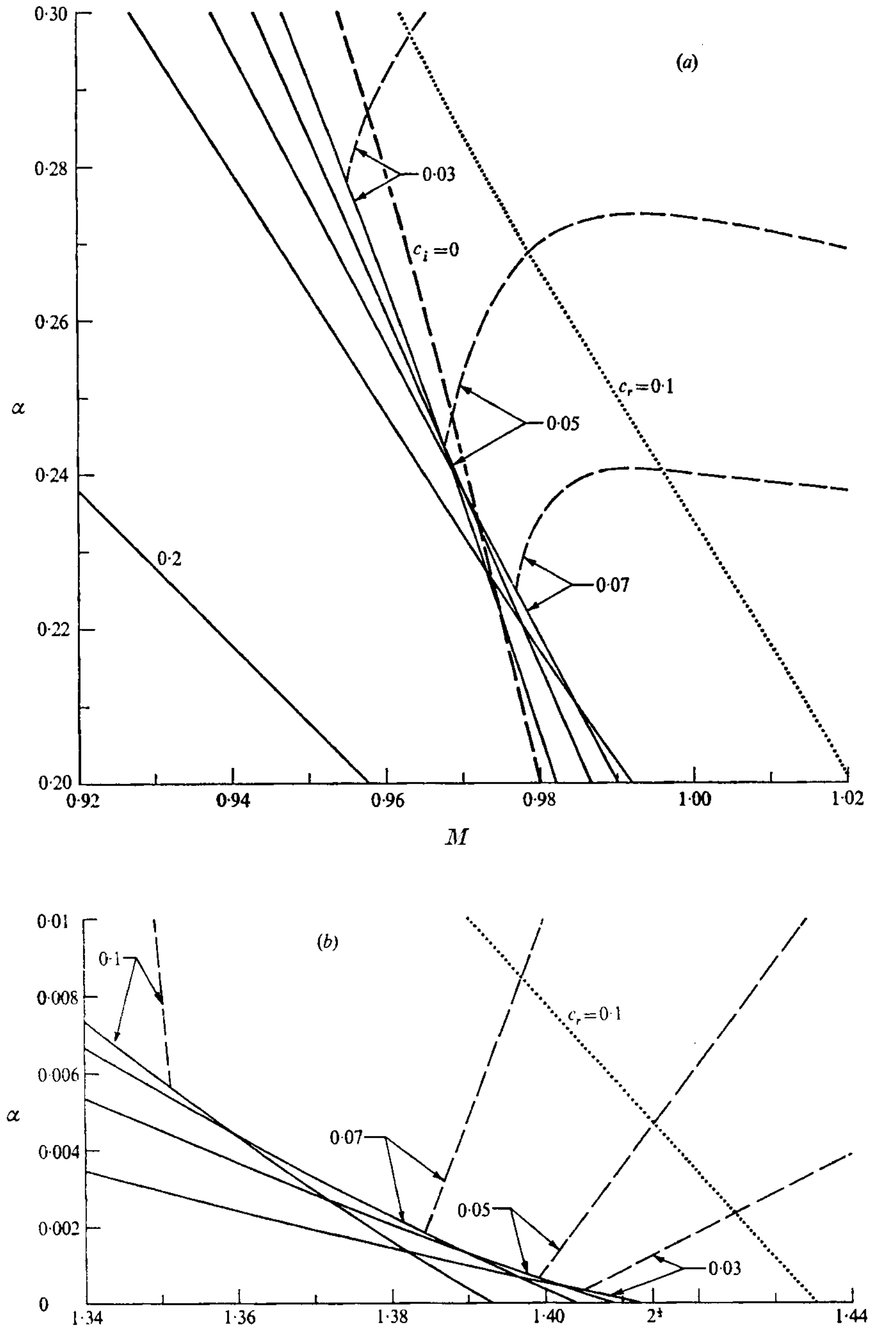

FIGURE 5. Enlargements of parts of the unstable region in the α, M plane. The marginal-stability curve is denoted by $c_i = 0$. ——, isolines of c_i in the region where the unstable waves are stationary ($c_r = 0$); — —, isolines of c_i where $c_r \neq 0$; ······, isoline $c_r = 0·1$.

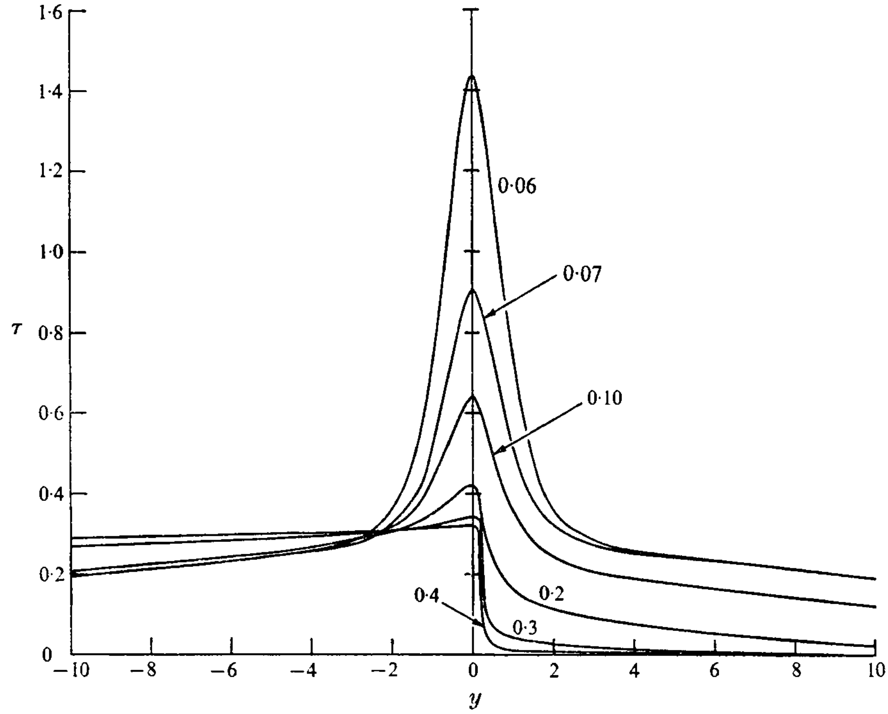

FIGURE 6. Reynolds stress τ as a function of y, along $M = 1{\cdot}14$, for indicated values of wavenumber α.

profile (4) there is only one critical layer. For the stationary modes, with $c_r = 0$, the critical layer is at $y = 0$; further, in the limit as $c_i \rightarrow 0$ we see that $p \rightarrow 0$ as $y \rightarrow \pm\infty$; therefore $\tau \rightarrow 0$ as $c_i \rightarrow 0$ for fixed $y \neq 0$. For the travelling modes, with $c_r \neq 0$, the critical layer is at $y = \tanh^{-1} c$; further, in the limit as $c_i \rightarrow 0$ we see that $p \rightarrow 0$ as $y \rightarrow (\operatorname{sgn} c_r)\infty$ but that p merely satisfies a radiation condition as $y \rightarrow -(\operatorname{sgn} c_r)\infty$; therefore τ is a non-zero constant on the radiating side of the critical layer and $\tau = 0$ on the other. These results provide a check on the accuracy of the numerical results as the stability boundary is approached.

4. Perturbation formulae

Perturbation of a known eigensolution

Let us suppose that somehow we know that the eigensolution

$$c = c_0, \quad \alpha^2 = \alpha_0^2, \quad M^2 = M_0^2, \quad p = p_0(y) \tag{9}$$

satisfies the problem (2) with (3) for some given function $U(y)$. To perturb this eigensolution, suppose that $c = c_0 + \delta c$ is an eigenvalue when

$$\alpha^2 = \alpha_0^2 + \delta\alpha^2, \quad M^2 = M_0^2 + \delta M^2, \tag{10}$$

for infinitesimals $\delta\alpha^2$ and δM^2. Then, assuming that the perturbation is a regular one, we deduce (e.g. following the Tollmien–Lin method described by Drazin & Howard 1966, p. 13) that

$$\delta c = I_1/I_0, \tag{11}$$

where the integrals I_0 and I_1 are defined by

$$I_0 = 2\int_{-\infty}^{\infty} (U-c_0)^{-3}(p_0'^2+\alpha_0^2 p_0^2)\,dy, \tag{12}$$

$$I_1 = \int_{-\infty}^{\infty} \{\alpha_0^2\delta M^2 + \delta\alpha^2[M_0^2-(U-c_0)^{-2}]\}p_0^2 dy. \tag{13}$$

In some special cases formula (11) is equivalent to formula (67) of Lin (1953).

For the hyperbolic-tangent profile (4), we take the known solution (9) to be (5). First we readily deduce that

$$I_0 = 2\alpha_0^2\pi i,$$

on taking the limit as $c_i \downarrow 0$ to resolve the singularity in the usual way. Similarly, we find

$$\begin{aligned} I_1 &= \int_{-1}^{1} \{\alpha_0^2\delta M^2+\delta\alpha^2[M_0^2-U^{-2}]\}(1-U^2)^{\alpha_0^2-1}dU \\ &= (\alpha_0^2\delta M^2+M_0^2\delta\alpha^2)\pi^{\frac{1}{2}}\Gamma(\alpha_0^2)/\Gamma(\alpha_0^2+\tfrac{1}{2})+\delta\alpha^2\left\{2+\int_{-1}^{1}[1-(1-U^2)^{\alpha_0^2-1}]\,U^{-2}dU\right\} \\ &= \alpha_0^2\pi^{\frac{1}{2}}\Gamma(\alpha_0^2)(\delta\alpha^2+\delta M^2)/\Gamma(\alpha_0^2+\tfrac{1}{2}), \end{aligned}$$

and hence that, when $\delta c_i > 0$,

$$\delta c = -\frac{i}{2\pi^{\frac{1}{2}}}\frac{\Gamma(\alpha_0^2)}{\Gamma(\alpha_0^2+\frac{1}{2})}(\delta\alpha^2+\delta M^2). \tag{14}$$

Formula (14) agrees with the result of the Tollmien–Lin formula for incompressible fluid with $M_0^2 = 0$, $\alpha_0^2 = 1$ and $\delta M^2 = 0$ (cf. Drazin & Howard 1966, p. 42). It agrees well also with our numerical results of § 3 when α_0 is greater than about 0·4, in view of the limitations of the formula (when $\delta\alpha^2$ and δM^2 are not infinitesimal) and of the numerical results (when $\delta\alpha^2$ and δM^2 are so small that (2) is nearly singular). However, formula (14) gives instability just inside, but not outside, the circle $\alpha^2+M^2 = 1$, in contradiction with our numerical results when α is less than about 0·37. This contradiction is reminiscent of one found by Huppert (1973) in comparing Howard's perturbation formula with direct numerical results for stability of a stratified incompressible fluid. The explanation of this apparent contradiction is not clear, but may be due to the fact that δc depends upon $\delta\alpha^2$ and δM^2 transcendentally, not linearly as we assumed in our derivation of formula (14). If δc were to depend upon $\delta\alpha^2$ and δM^2 algebraically, but not linearly, one would expect the integral I_0 to vanish (Banks & Drazin 1973, § 5).

The long-wave approximation

Gill & Drazin (1965) applied methods developed by Drazin & Howard (1966) to the solution of the problem (2) with (3) for small α. Their results may be extended further, and here it suffices to state the new results we wish to use, without dwelling upon their derivation.

If there exist $U_{\pm\infty} = \lim_{y\to\pm\infty} U(y)$, then the eigenfunctions of the problem (2) with (3) can be expressed in the form

$$p(y) = \begin{cases} K_+P_+(y)\exp(-\alpha l_+ y) & \text{for} \quad y \geqslant 0, \\ K_-P_-(y)\exp(\alpha l_- y) & \text{for} \quad y \leqslant 0, \end{cases} \tag{15}$$

where $K_{\pm}$ are some constants, $l_{\pm}^2 = 1-M^2(U_{\pm\infty}-c)^2$, and the roots $l_{\pm}$ are chosen such that $\mathrm{Re}\, l_{\pm} > 0$ or such that $\mathrm{Re}\, l_{\pm} = 0$ and the radiation conditions at infinity are satisfied. Further we may take the expansion

$$P_{\pm}(y) = P_{\pm 0}(y) + \alpha P_{\pm 1}(y) + \alpha^2 P_{\pm 2}(y) + \dots \quad \text{as} \quad \alpha \to 0, \tag{16}$$

where

$$P_{\pm 0}(y) = 1, \tag{17a}$$

$$P_{\pm 1}(y) = \pm l_{\pm} \int_{\pm\infty}^{y} \left(1 - \frac{(U_1-c)^2}{(U_{\pm\infty}-c)^2}\right) dy_1, \tag{17b}$$

$$P_{\pm 2}(y) = \int_{\pm\infty}^{y} \left\{ l_{\pm}^2 \int_{\pm\infty}^{y_1} \left(1 - \frac{(U_2-c)^2}{(U_{\pm\infty}-c)^2}\right) dy_2 + (U_1-c)^2 \int_{\pm\infty}^{y_1} [(U_2-c)^{-2} - (U_{\pm\infty}-c)^{-2}]\, dy_2 \right\} dy_1, \tag{17c}$$

etc. Then it can be shown that expansion of the eigenvalue relation gives

$$\frac{l_+}{(U_\infty-c)^2} + \frac{l_-}{(U_{-\infty}-c)^2} + \alpha \left\{ \int_0^\infty [(U-c)^{-2} - (U_\infty-c)^{-2}]\, dy + \int_{-\infty}^{0} [(U-c)^{-2} - (U_{-\infty}-c)^{-2}]\, dy - l_+ l_- (U_\infty-c)^{-2} \int_{-\infty}^{0} \left(1 - \frac{(U-c)^2}{(U_{-\infty}-c)^2}\right) dy - l_+ l_- (U_{-\infty}-c)^{-2} \int_0^\infty \left(1 - \frac{(U-c)^2}{(U_\infty-c)^2}\right) dy \right\} + \dots = 0 \quad \text{as} \quad \alpha \to 0. \tag{18}$$

For jets, formula (18) agrees with formula (4) of Gill & Drazin (1965), who went on to find the stability characteristics of long waves.

For our hyperbolic-tangent shear layer (4), it can be shown to give

$$(1+c)^2\{1-M^2(1-c)^2\}^{\frac{1}{2}} + (1-c)^2\{1-M^2(1+c)^2\} - 2\alpha[\{1-M^2(1-c)^2\}^{\frac{1}{2}} \times \{1-M^2(1+c)^2\}^{\frac{1}{2}} + 1 + c\log\{(c+1)/(c-1)\}] + \dots = 0. \tag{19}$$

Ignoring the terms in powers of α, we recover the eigensolution (7) for a vortex sheet, which gives $c = \pm c_v$, say. Using formula (19) to perturb these values, we find at length that

$$c = c_v + \alpha \frac{1+c_v}{1-c_v} \{1-M^2(1-c_v)^2\}^{\frac{1}{2}} \frac{2 + c_v \log\{(c_v+1)/(c_v-1)\}}{1+4M^2-3(1+4M^2)^{\frac{1}{2}}} + O(\alpha^2)$$

$$\text{as } \alpha \to 0 \text{ for fixed } M^2 \neq 2. \tag{20}$$

This gives unstable modes with

$$c = c_v + i\alpha \frac{1+c_v}{1-c_v} \{M^2(1-c_v)^2 - 1\}^{\frac{1}{2}} \frac{2 + c_v [\log\{(1+c_v)/(1-c_v)\} - \pi i]}{1+4M^2-3(1+4M^2)^{\frac{1}{2}}} + O(\alpha^2)$$

$$\text{as } \alpha \to 0 \text{ for fixed } M^2 > 2. \tag{21}$$

Formula (21) shows that the modes with $c = c_v$ (or $c = -c_v$, similarly) are stable for $\alpha = 0$ but unstable for small positive α. Thus there is instability just above the M axis for $M^2 > 2$, and hence there is no critical value of the Mach number above which two-dimensional waves are stable.

Ensuring again that the square-roots in formula (19) have non-negative real parts, we can find a second unstable mode, for which we have

$$c \sim i\alpha M^2 (M^2-1)^{\frac{1}{2}}/(2-M^2) \quad \text{as } \alpha \to 0 \text{ for } 1 < M^2 < 2; \tag{22}$$

this corresponds to the Blumen solution (5) in the limit as $\alpha \to 0$ and to the degenerate solution (8) for the vortex sheet. Using formula (19) to investigate the stability characteristics when both α and $M^2 - 2$ are small, we find that for unstable modes

$$6c^3 + (2 - M^2)c - 2i\alpha = O(\alpha^2, c(2-M^2)^2) \quad \text{as} \quad \alpha \to 0, \quad M^2 \to 2. \tag{23}$$

This cubic approximation gives two admissible unstable roots c and one inadmissible stable root. The two unstable roots are pure imaginary, $c = ic_i$, when $M^2 < 2 - 3(6\alpha^2)^{\frac{1}{3}}$, but complex, $c = \pm c_r + ic_i$, when $M^2 > 2 - 3(6\alpha^2)^{\frac{1}{3}}$. Formulae (22) and (23) agree well with the numerical results of §3.

5. Discussion

We have found that the hyperbolic-tangent shear layer is unstable to two-dimensional disturbances at each value of the Mach number, however large. This follows from an asymptotic formula and was confirmed numerically. It is plausible that this result is typical of smoothly varying shear layers. It is contrary to earlier results found for the discontinuous vortex sheet, and so casts doubt on the physical value of models which incorporate a vortex sheet.

The difference between stability characteristics of long waves on the shear layer and those on the vortex sheet is chiefly due to the occurrence of a second mode for the shear layer which is degenerate for the vortex sheet. This second mode has been elucidated by our asymptotic and numerical analysis, it being seen to occur chiefly when the Mach number is greater than one. Also the principle of exchange of stabilities was found to be invalid where this second mode occurs, although it is valid for the vortex sheet.

Our computations show that the relative growth rate αc_i of the second mode is always small, typically one order of magnitude less than that of the first mode (see figure 2). However, the spatial as well as the temporal growth of a mode is important in practice. Now (2) has solutions which behave like

$$p \sim \text{constant} \times \exp[-\alpha\{1 - M^2(1 \mp c)^2\}^{\frac{1}{2}}|y|] \quad \text{as} \quad y \to \pm\infty.$$

Thus a neutral mode which is subsonic relative to the basic flow at infinity decays exponentially with distance $|y|$ from the shear layer, but one which is supersonic does not decay at all. In fact a neutral mode which is supersonic relative to the basic flow at infinity radiates outwards like a sound wave, unattenuated in this two-dimensional model. The unstable supersonic modes do decay exponentially with $|y|$, but much more slowly than the subsonic modes. It follows that the second mode, though of slow temporal growth, is likely to be of greater amplitude than the first mode far away from the shear layer whose instability generates the modes.

In fact each supersonic mode is found to be like a sound wave, with velocity $\pm M^{-1}$ relative to the basic stream, whose velocity approaches ± 1 as $y \to \pm\infty$; thus $c = \pm 1 \mp M^{-1} + o(M^{-1})$ as $M \to \infty$. This is consistent with Lin's (1953) deduction that each neutral mode is not supersonic relative to either stream.

This discussion warns against the thoughtless use of vortex sheets, although they have the great advantage of mathematical simplicity. Whether in aerodynamic noise theory or in physically distinct but mathematically analogous fields such as shallow-water waves, it may be undesirable to use a model which renders degenerate an important physical class of modes.

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